MATH 521A: Abstract Algebra
Preparation for Exam 3

1. Set $f(x) = x, g(x) = x + 2x^2$, both in $\mathbb{Z}_4[x]$. Prove that $f|g$ and $g|f$.

2. Set $f(x) = x + a + b$, $g(x) = x^3 - 3abx + a^3 + b^3$, both in $\mathbb{Q}[x]$. Find $\gcd(f, g)$.

3. Let R be a ring. Characterize all polynomials $f, g \in R[x]$ such that $\deg(f + g) < \max(\deg(f), \deg(g))$.

4. In $F[x]$, prove that "is an associate of" is an equivalence relation.

5. Set $f(x) = x^n \in F[x]$. Carefully determine all divisors of $f(x)$.

6. Let $f(x) \in \mathbb{Z}[x]$ be monic. Suppose that $a \in \mathbb{Q}$ and $f(a) = 0$. Prove that, in fact, $a \in \mathbb{Z}$.

7. Set $f(x) = 3x^3 + 5x^2 + 6x$, $g(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$, both in $\mathbb{Z}_7[x]$. Use the extended Euclidean algorithm to find $\gcd(f, g)$ and to find polynomials $s(x), t(x)$ such that $\gcd(f, g) = f(x)s(x) + g(x)t(x)$.

8. Determine for which a the polynomial $f(x) = x^3 + ax^2 - ax + 1 \in \mathbb{Z}_7[x]$ is irreducible.

9. Factor $f(x) = x^4 + x^3 + 6x^2 - 14x + 16 \in \mathbb{Q}[x]$ into irreducibles.

10. Let p be prime, and consider the polynomial $f(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$. Prove that $f(x)$ is irreducible. Hint: You may use without proof the fact that p divides $\binom{p}{a}$ for any a with $1 \leq a \leq p - 1$.

11. Let F be a field. We define the "derivative" operator $D : F[x] \to F[x]$ via

$$D(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 1a_1.$$

Prove that this operator satisfies, for all $f, g \in F[x]$ and for all $c \in F$:
(a) $D(f + g) = D(f) + D(g)$; (b) $D(cf) = cD(f)$; (c) $D(fg) = fD(g) + D(f)g$

12. Let F, D be as in Problem 11. Suppose $f, g \in F[x]$ and $f^2|g$. Prove that $f|D(g)$. What does this mean if f is a linear polynomial?

13. Let $f(x) \in F[x]$ be irreducible. Suppose that $f(x)|g_1(x)g_2(x)\cdots g_k(x)$. Prove that for some i with $1 \leq i \leq k$, we have $f(x)|g_i(x)$.

14. Let R, S be rings and $\phi : R \to S$ a ring homomorphism. Define $\tau : R[x] \to S[x]$ via

$$\tau(a_n x^n + \cdots + a_1 x + a_0) = \phi(a_n)x^n + \cdots + \phi(a_1)x + \phi(a_0).$$

Prove that τ is a ring homomorphism.

15. For ring R, $a \in R$, and $n \in \mathbb{N}$, we say a has *additive order* n if $\underbrace{a + a + \cdots + a}_{n} = 0_R$, and for $m < n$ we have $\underbrace{a + a + \cdots + a}_{m} \neq 0_R$. We write this $ord_R(a) = n$. Suppose every element of R has an order (not necessarily the same one). Prove that every element of $R[x]$ has an order.